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Laminar Viscous Effects over Blunt Cones at Hypersonic Conditions

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Nomenclature

C_{D_f} = friction-drag coefficient referenced to base area
M = Mach number
r_b, r_n = base and nose radii, respectively
U_∞ = freestream velocity
ϵ = $(\mu^*(U_\infty{}^{*2}/C_p{}^*)/\rho_\infty{}^* U_\infty{}^* r_n{}^*)^{1/2}$, Van Dyke's expansion parameter

Subscripts

0 = stagnation conditions
STJ = slip and temperature jump effect
TVC = transverse curvature effect
vort = vorticity effect
w = wall
* = dimensional quantity

SPHERICALLY blunted cones have been used extensively in experimental studies at supersonic and hypersonic conditions. Since classical laminar boundary-layer theory has not been successful under moderately low Reynolds number conditions, a need has existed for several years for a theoretical model to explain the experimental observations such as pressure and heat-transfer distributions over bodies and prediction of zero-lift drag. It turns out that the skin-friction drag is more sensitive to low Reynolds number viscous effects than are the pressure and heat-transfer distributions. Also the prediction of zero-lift drag is important in the analysis of wind-tunnel data.

Theoretical analysis of viscous effects at hypersonic conditions can proceed in either of two ways. Early extensions of classical boundary-layer theory considered the effects of longitudinal body curvature (LC). More recently the separate effects of transverse curvature (TVC), and slip and temperature jump (STJ) were considered. The last effect to receive attention was shock generated external vorticity (vort) in the stagnation region of blunt bodies. These effects are considered by including the next order of terms in the boundary-layer equations (LC and TVC) and by modifying the wall and outer edge boundary conditions (STJ and vort). The remaining effect of the same order of magnitude but not mentioned previously is boundary-layer displacement effect (disp) which is a global effect and must be considered with the entire external inviscid flowfield. The approach described previously is called herein a first-order treatment of higher-order boundary-layer effects.

Another treatment of higher-order effects is based upon a perturbation expansion of the variables in the Navier-Stokes equations and the method of matched asymptotic expansions. Retaining first-order terms gives the classical Prandtl boundary-layer equations. Second-order terms include all the effects described in first-order treatment.

The primary difference between the first- and second-order treatments of higher-order effects is that the higher-order effects in second-order theory are with one exception linearly independent whereas the effects in the first-order treatment are coupled, nonlinear effects. It is therefore of interest to consider comparison of the two theories with each other as well as comparison of each with experimental data.

Experimental data from wind-tunnel tests have provided the incentive for a study of higher-order boundary-layer effects under nearly perfect gas conditions. Several years ago the strong influence of higher-order viscous effects was experimentally observed on the drag of slender cones at $M_\infty = 10$ to 20.[1] Since that time a study of theoretical and numerical methods has been made to analyze and predict observed experimental trends. To date the best available theoretical models and numerical methods have not been successful in predicting the observed results over the entire ranges of Mach and Reynolds numbers experimentally studied.

The purpose of the present paper is to indicate the results of the application of first- and second-order boundary-layer theories to a sphere-cone for a range of Reynolds numbers at $M_\infty = 18$, and comparisons with experimental data indicate where one might expect the theoretical models to be applicable.

Lewis and Whitfield[1] presented some of the early work done in von Karman Gas Dynamics Facility (VKF) where they applied iterated inviscid-viscous flowfields models to predict pressure and heat-transfer distributions and zero-lift drag of a 9° half-angle, spherically blunted cone at $M_\infty = 18$. In that work an inverse blunt body and characteristics solution for the inviscid outer flow was iterated with a first-order boundary-layer solution which included approximate transverse curvature terms. The blunt body and characteristics method used was due to Inouye, Rakich, and Lomax[2] and the boundary-layer method was that of Clutter and Smith.[3] In many respects the results of the predictions of Lewis and Whitfield were in surprisingly good agreement with the experimental results since the effects of shock-generated external vorticity and slip and temperature jump were not considered and the effects of transverse curvature and displacement were only approximately treated.

Davis and Flügge-Lotz[4] considered second-order boundary-layer effects on hyperboloids, paraboloids, and spheres at infinite Mach number and ten, respectively. The theory of Van Dyke[5] was used with an implicit finite-difference scheme originally proposed by Flügge-Lotz and Blottner[6] for treating the classical first-order boundary-layer equations for two-dimensional flows. As will be shown in this paper, the theory of Van Dyke when coupled with the implicit finite-difference method of Davis and Flügge-Lotz gives a powerful tool for extending classical boundary-layer theory to lower Reynolds number.

In addition to the second-order treatment based on Van Dyke-Davis and Flügge-Lotz, a first-order treatment of vorticity, displacement, transverse curvature (TVC), and slip and temperature jump (STJ) was developed by the author[7] based on a modification of the first-order boundary-layer method of Clutter and Smith. The treatment of vorticity is based on the suggestion of Hayes and Probstein[8] where the outer boundary condition is changed to account for an increase in velocity and a nonzero velocity gradient.

Second-order boundary-layer theory is discussed in Refs. 7 and 9-12. The numerical methods used to compute the

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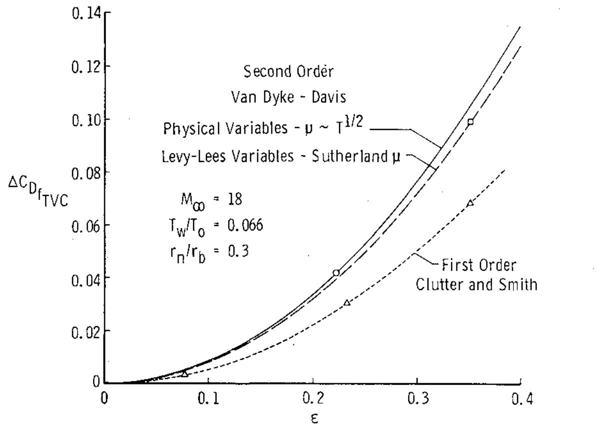

Fig. 1 Transverse-curvature-induced friction drag from first- and second-order theories.

second-order effects shown in this paper are described in detail in Refs. 10 and 11.

First-Order Effects

The original treatment of Clutter and Smith[3] included an error in the transverse curvature terms arising from the transformations and evaluation of the stream function. The error would be difficult to correct without a reformulation in terms of a more appropriate set of transformations. In the revised version of the report, Clutter and Smith termed the transverse curvature as approximate (which indeed it must be), and the degree of approximation was never determined to this writer's knowledge. Because of the extensive use which has been made of the various forms of the Douglas programs, some comparisons will be given here to illustrate the degree of approximation which one might expect from using the so-called "approximate" Clutter and Smith formulation.

A new and correct formulation of the first-order transverse curvature effect has recently been given by Jaffe, Lind, and Smith.[13] In this more recent work, the Levy-Lees and Probstein and Elliott transformations were combined so that the resulting transformed momentum and energy-equations are quite similar in form to the Clutter and Smith equations with additional terms.

First-order treatment of vorticity interaction, slip and temperature jump and displacement in addition to transverse curvature is described in detail in Ref. 7.

A comparison of the transverse-curvature-induced friction drag at $M_\infty = 18$ is shown in Fig. 1. The differences are small between second-order solutions with different viscosity laws; however, the approximate treatment of Clutter and Smith predicts an increment about 30% less. This is a significant error, and the Clutter and Smith treatment cannot be recommended. As noted earlier, TVC was correctly treated by Jaffe, Lind, and Smith.[13]

One of the most instructive comparisons between first- and second-order numerical results and experimental data is given in Fig. 2. The total drag predicted by the two treatments is compared with the experimental data of Whitfield and Griffith.[14] For $M_\infty \simeq 18$ and $\epsilon < 0.15$ we find good agreement between the numerical results and the available experimental data. For $\epsilon > 0.2$ the agreement is poor. If we recall that in the second-order theory[5] it was assumed that $\epsilon \ll 1$, one should expect that as ϵ increases at some value the theory will no longer be applicable. For these conditions, this point appears to be near $\epsilon = 0.15$.

The previous first-order results of Lewis and Whitfield[1] are also shown for comparison. The apparently good agreement is simply fortuitous since only approximate transverse curvature (Clutter and Smith) and displacement (iterated inviscid-viscous flowfields) were included. The comparisons of transverse-curvature-induced friction drag showed that the

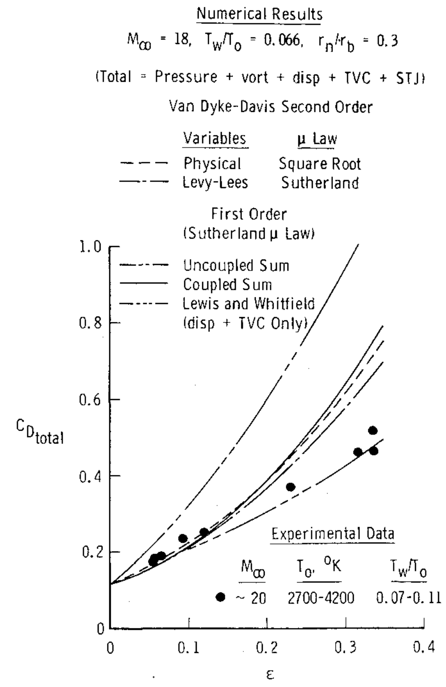

Fig. 2 Heat transfer to a spherically blunted cone at $M_\infty \simeq 18$.

approximate Clutter and Smith treatment led to an error of about 30% in that component. Also, iteration of the inviscid-viscous flowfields reduced the displacement-induced pressure and friction drag.

Finally, the data show that for $\epsilon < 0.15$ first- and second-order treatments are in substantial agreement with the experimental zero-lift drag data. For $\epsilon > 0.2$ the predictions were substantially in error. From these comparisons it is clear that higher-order boundary-layer treatments are not the correct approach where $\epsilon > 0.2$. A theoretical model is needed which properly takes into account all higher-order effects such as a viscous "external" flow and the effects of transport properties on the shock wave.

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Method of Weighted Residuals Applied to Free Shear Layers

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THE purpose of this Note is to present a detailed application of the method of weighted residuals (MWR) to laminar incompressible free shear layers. In the past, the MWR has been applied to laminar boundary-layer flows with and without suction and blowing,[1–3] and to turbulent boundary layers.[4] These previous works have all been oriented to attached boundary layers and not to free shear layers or jets. It was decided to investigate the laminar free shear layer as an initial step in a study of turbulent jets by the MWR since an exact solution is available[5] and since the physics of the problem is simple, one is allowed to easily examine the disadvantages of the MWR.

One of the particular questions to be investigated concerns the choice of the velocity profile polynomial and the weighting function, both of which are used in the integral method. A certain arbitrariness is present in the MWR; specifically, Bethel[1,3] uses several different forms of the velocity profile depending on the external flowfield. Unfortunately, there is no way to determine beforehand what is the proper choice of approximating function.

The equations to be solved are the same as those given by Bethel[1] with the exception that the pressure gradient parameter $\dot{U}/U = 0$. Thus

$$\frac{d}{d\xi}\int_{-\infty}^{+\infty} h_i(u)u\,d\eta - \int_{-\infty}^{+\infty} h_i'(u)\frac{\partial^2 u}{\partial \eta^2}\,d\eta = 0 \qquad i = 1,2,\ldots N$$

where $\xi \equiv x/L$, $\eta \equiv Re_\infty^{1/2}y/L$, $u \equiv \bar{u}/u_\infty$. At this point, for the free shear layer problem, we restrict $h_i(u)$ to be lim $\eta \to -\infty$, $h_i(u) = 0$. We then make the usual change of variable $\theta(\xi,u) = 1/(\partial u/\partial\eta)$ and integrate the second term by parts to obtain

$$\frac{d}{d\xi}\int_0^1 h_i(u)u\theta du + \int_0^1 h_i''(u)\,\frac{1}{\theta}\,du = 0 \qquad i = 1,2,\ldots N$$

where θ is some appropriate form of inverse shear stress profile and $h_i(u)$ is an as yet unspecified weighting function.

The boundary conditions on the shear stress dictate that $\theta \to \infty$ as $u \to 0$ and as $u \to 1$; i.e., there are poles at $u = 0$ and 1. The simplest form of an assumed shear-stress distribution is $\theta(\xi,u) \propto 1/[u(1-u)]$. Admittedly, there are many other forms of θ which will satisfy the required conditions and this is the first point where a decision must be made. We adhere to the belief that one should always try the simplest case that meets all required conditions. Of course, if this approach fails, perhaps one should re-examine the required conditions—they may be incomplete. Since the problem admits to a similar solution and there is an exact solution of this form available,[5] we follow Bethel[1] in saying

$$\theta(\xi,u) = \frac{\xi^{1/2}}{u(1-u)}\sum_{j=1}^{N} c_j u^{j-1}$$

where the c_j are the unknown constants.

The second decision to be made in the solution is what should be the form of the weighting function. The simplest general choice appears to be

$$h_i(u) = u^{mi+n}(1-u)^{ki+l}$$

where m,n,k, and l are constant integers. Bethel[1] points out in the boundary-layer problem that if $h_i(u) = (1-u)^i$ is used and that if $i = 1$, then the resulting integral equation is the Karman-Pohlhausen equation. This occurrence at least assures one that he is using the conservation relations in the solution of the problem. No such occurrence is possible in the free shear layer problem because the integral momentum equation has to be employed as an auxiliary equation to determine the position of the dividing stream line; it can not be used in the system of equations employed to find the unknown c_j. Thus the system of equations contains arbitrary moments of the velocity profile unrelated to physical quantities. For this reason, it seems unwise to try any more complicated weighting function than that suggested before, at least for the present. The one task that the suggested form of $h_i(u)$ can accomplish is to simplify the integrals that have to be carried out. So, with ease of integration as a goal, the final form of the algebraic equations becomes

$$\sum_1^N c_j\Gamma(mi+j+n)\Big/\Gamma(mi+ki+j+n+l) + 2(ki+l)\sum_1^N b_j\Gamma(mi+n+j-1)\Big/\Gamma(mi+ki+j+n+l+1)\times[-(2m+k)ij+kij^2+(1-l-2n)j+(l-1)j^2] = 0 \qquad i = 1,2,\ldots N$$

where Γ indicates the gamma function and where the b_j term arises from the same substitution employed by Bethel.[1] The collocation scheme for relating the b_j to the c_j is

$$u_p = (p-1)/(N-C),\ p = 1,2,\ldots N,\ 0 \leqslant C \leqslant 1$$

The constant C is arbitrary except as specified and serves to alter the collocation points. This parameter has been inserted in order to study the effect of the collocation scheme on the convergence of the solution.

The solution of the free shear layer problem would not be complete without obtaining the velocity profiles of u and v in terms of the physical plane coordinates. The procedure

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